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On Jets Produced by Conical Nozzles

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by Ignace I. Kolodner

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When a steady stream of fluid is ejected from a conical nozzle, a jet forms having the shape of a hollow tulip attached to the orifice of the nozzle. It is also observed that the jet will oscillate about its equilibrium configuration, unless it is vented by, say, piercing it at some point.

These observations can be explained qualitatively as follows. The tulip shape of the jet results from the action of surface tension which tends to contract the jet toward the axis of the nozzle. Thus, for a given set of data there forms an equilibrium jet envelope. If for some reason the jet envelope is made to deviate initially from its equilibrium shape and if it is closed (i.e., unvented), oscillations will be induced. Indeed, a contraction of the jet compresses the air enclosed by it and thus produces an outward force which would reexpand the jet and cause it to overshoot its equilibrium position, after which it would be stopped and pushed in again. If the jet is vented no oscillations will occur, since the inside air, being kept at fixed pressure, would not supply an outward force. The main point to be stressed here is that surface tension alone could not account for the appearance of oscillations. Of course, a vented contracted jet will resume its equilibrium shape because the new supply of liquid from the nozzle will essentially follow the equilibrium path, while the fluid in the contracted portion will flow out.

The object of this paper is the quantitative study of these phenomena carried out under the assumption of the thin jet theory. In section I the mathematical problem is formulated and equations of motion of the fluid ejected from

the nozzle are derived. In view of the intended study of oscillations, the general case of unsteady motion is considered. Section II contains the study of the equilibrium jet. If gravity effects are neglected, the solution can be obtained in a closed form. Corrections due to gravity are also estimated. In section III, results of section II are discussed and compared with some experimental data. Section IV contains a study of the unsteady jet. In this case, explicit results cannot be derived, nor are they desired due to the incertitude about the initial jet shape, but a satisfactory approximate treatment is obtained.

As to the equilibrium jet, our results agree very well with the available experimental data, but only after corrections due to gravity are taken into account. See fig. 4. It is rather surprising that gravity ought to have a detectable effect. The thin jet theory leads to satisfactory results only if the mass flux exceeds a certain critical value. Below this critical value, the mathematical problem for the equilibrium jet fails to have a solution, showing that in this case the thin jet assumption does not apply. Concerning the jet oscillations, we find that they will be damped. The frequency of these oscillations is found to be of the order of 2,000 cycles per second when the outside pressure is normal. If the pressure is reduced this frequency may drop to as low as 20-25 cycles per second. The time in which the amplitude of oscillations will be reduced to half its original value is found to be of the order of .03 second and is independent of pressure.

Other important features observed in connection with conical jets are:

1. slow period oscillations of the jet envelope,
2. shattering (or dropletization) when the mass flux exceeds a certain critical value.

The theory of the first of these could be obtained from the general equations derived in section I provided that

one has a satisfactory mechanism for the production of a long period phenomenon. It has been suggested that air may diffuse through the envelope, thus resulting in a different volume-pressure relation for the air enclosed by the jet than the one considered in section IV. This would result in a different formula for the period, but the same expression for the half-time of such oscillations. As to the shattering, the techniques employed in this paper are not adequate to consider this phase. A study of it is now in progress and will be discussed in a future report.

The author has been unable to find in the literature any reference to unsteady conical jets. As to the steady case, this has been studied by T. G. Hodgkinson, "Control by Surface Tension of a Conical Fluid Sheet Jet", Porton Technical Paper No. 174, May 1950. In our view, Hodgkinson's treatment is inconsistent with the assumptions he makes.

The author wishes to thank Mr. John S. Derr, Jr. for supplying him with the experimental results used in this paper.

I. Formulation of the problem.

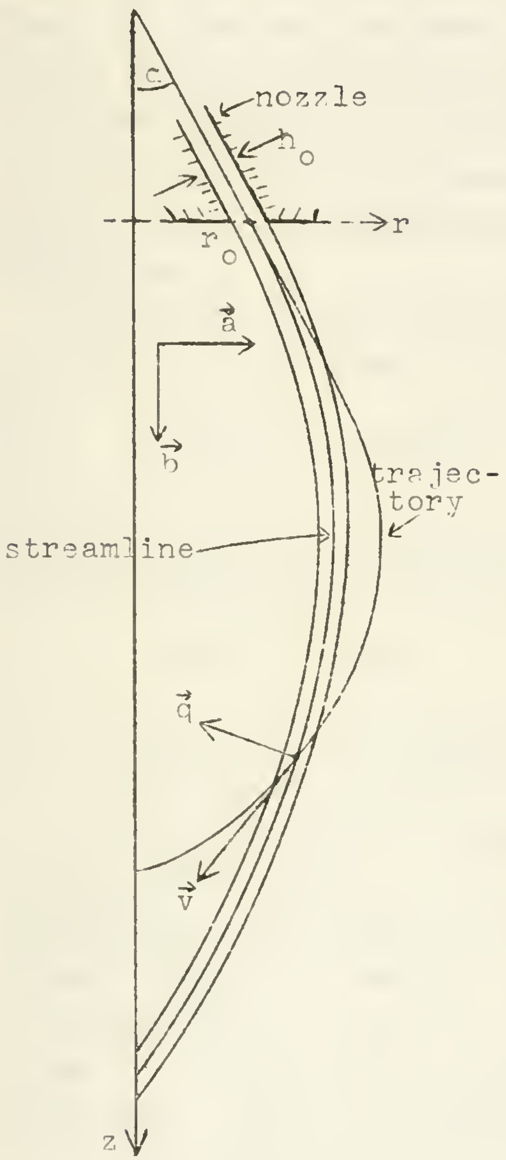

Figure 1.

Denote by r_o, h_o, α the nozzle mean radius at orifice, nozzle thickness, and nozzle angle, respectively. See fig. 1. We consider the flow of liquid emerging from the nozzle with velocity v_o. If ρ is the density of the liquid then

$$(1) \qquad M = 2\pi\rho\, r_o h_o v_o$$

is the mass flux through orifice.

We assume that the nozzle axis is vertical so that even if gravity effects are considered, the axial symmetry of the problem is preserved. It is therefore sufficient to confine our attention to the meridional section of the jet. Thus we consider the liquid contained between two axial planes making the angle $\Delta\varphi$, see Fig. 1.

We use the Lagrangian description of the liquid, that is, consider particle paths which will be called trajectories. What one actually observes is not the trajectories of individual particles, but the flow pattern at a fixed time, which may be described in terms of streamsurfaces, the outer streamsurfaces being the boundaries of the jet. We shall call streamline the meridional section of the median streamsurface.

Under the thin jet assumption it is assumed that the velocity field is constant along each normal to the streamline. It then follows that the thin jet is described completely by the streamline, and that the trajectories of particles leaving the orifice at the same time are identical. The thickness of the jet measured along the normal to the streamline can be determined from the incompressibility assumption.

Denote by $\vec{x}(\tau,t)$ the position at time t of a particle which at time τ was at the orifice. Thus

$$(2) \qquad \vec{x}(\tau,t) = r(\tau,t)\vec{a} + z(\tau,t)\vec{b}$$

where $\vec{a}$ and $\vec{b}$ are unit vectors in the radial and vertical (downwards) directions, respectively, and

$$(3) \qquad \vec{x}(\tau,\tau) = r_o\vec{a}.$$

For a fixed τ, $\vec{x}$ is defined for $\tau \leq t \leq T(\tau)$, $T(\tau)$ being the time at which this particle reaches the axis (i.e., $r(\tau, T(\tau)) = 0$), and $\vec{x}$ represents the trajectory of the particle labelled by τ. Let $\tau_o(t)$ be the time at which the particle that reaches the axis at time t, has passed through orifice. Then, for $\tau_o(t) \leq \tau \leq t$, and fixed t, $\vec{x}$ represents the totality of particles that are observed at time t, hence yields a parametric representation of the streamline. $\tau_o(t)$ is implicitly defined by $r(\tau_o(t),t) = 0$.

Consider the mass Δm of fluid that flowed through the orifice between times τ and $\tau + \Delta\tau$,

$$(4) \qquad \Delta m = \frac{\Delta\varphi}{2\pi} M \Delta\tau.$$

At time $t > \tau + \Delta\tau$, this mass occupies the position on the streamline between $\vec{x}(\tau + \Delta\tau, t)$ and $\vec{x}(\tau, t)$ and has an area

$$(5) \qquad \Delta s = r\Delta\psi \, |x_\tau| \, \Delta\tau$$

in common with the median streamsurface. The forces acting on Δm are:

1. force of inertia = $-\Delta m\vec{x}_{tt}$
2. force of gravity = $\Delta m g\vec{b}$

where g is the acceleration of gravity,

3. pressure force = $-\Delta p\ \Delta s\vec{q}$

where $\vec{q}$ is the unit normal vector to the streamsurface pointing inwards, and Δp is the excess of pressure of air enclosed by the jet over the outside air pressure,

4. force of tension = $2\Delta s T H\vec{q}$

where T is the coefficient of surface tension, and H is the sum of principal curvatures of the streamsurface. The Newton equations of motion for the mass Δm now yield on using (4) and (5),

$$M\vec{x}_{tt} = Mg\vec{b} + 2\pi r|x_\tau|(-\Delta p + 2TH)\vec{q}. \tag{6}$$

We have,

$$|x_\tau|\vec{q} = z_\tau\vec{a} - r_\tau\vec{b} \tag{7}$$

$$H = -|x_\tau|^{-3}(r_\tau z_{\tau\tau} - z_\tau r_{\tau\tau} + r^{-1}z_\tau|x_\tau|^2)\ ^{*} \tag{8}$$

and

$$\Delta p = P - p\ , \tag{9}$$

where p is the pressure of outside air (assumed atmospheric) and P is the pressure of air enclosed by the jet and is a function of the volume V enclosed by the jet. If the jet

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[*] The sign of H is so chosen that H is positive when the streamline is concave toward the axis, in which case the force of tension is directed inwards.

is either vented, or does not close, $\Delta p = 0$. Observing that

$$(10) \qquad V = -\pi \int_{\tau_o(t)}^{t} r^2 z_\tau d\tau = V(t),$$

we see that the equations of motion (6) form a set of two integro-partial-differential equations. These equations determine $\vec{x}$ provided that we specify the initial jet, that is, $\vec{x}(\tau,0)$, $\vec{x}_t(\tau,0)$, and the conditions at the orifice. The latter are given by equation (3), and the specification of velocities at the orifice is

$$(11) \qquad \vec{v}(\tau,\tau) = \vec{x}_t(\tau,\tau) = v_o \sin \alpha \, \vec{a} + v_o \cos \alpha \, \vec{b}.$$

II. Steady flow solution; case of the vented jet.

We have a steady flow if the trajectories of particles leaving the orifice at different times are identical. That is, for all u, τ, and t,

$$\vec{x}(\tau + u, t + u) \equiv \vec{x}(\tau, t).$$

Differentiating with respect to u, and setting $u = 0$ one gets

$$\vec{x}_\tau + \vec{x}_t = 0$$

and this shows that the flow is steady if $\vec{x}$ is a function of $t - \tau$ only. It follows then that the trajectory traces the streamline.

To find the steady flow streamline we thus assume that $\vec{x}(\tau, t) = \vec{x}(t - \tau)$, and use equations (I-6) in which $\vec{x}_\tau$, $\vec{x}_{\tau\tau}$ can be now replaced by $-\vec{x}_t$ and $\vec{x}_{tt}$ respectively. This equation will have solutions depending on $t - \tau$ only if Δp is also a function of $t - \tau$. Since Δp is a function of t and not of τ, see equation (I-9) and (I-10), it follows that Δp must be constant, which we assume.

Writing t in place of $t - \tau$, and introducing dimensionless variables defined by

$$(1) \qquad \begin{cases} \vec{x}(t) = r_o \vec{x}\,'(t') \\ t = r_o v_o^{-1} t' \end{cases}$$

the equations (I-6) now become (on using a dot to denote differentiation with respect to t' and dropping all primes),

$$(2) \qquad \ddot{\vec{x}} = \mu \vec{b} + (2\eta r - \nu |\dot{x}|^{-3} r[\dot{r}\ddot{z} - \dot{z}\ddot{r} + r^{-1}\dot{z}|\dot{x}|^2])(\dot{z}\vec{a} - \dot{r}\vec{b})$$

where

$$(3) \qquad \begin{cases} \mu = \dfrac{r_o g}{v_o^2} \\ \eta = \dfrac{\pi \Delta p r_o^2}{M v_o} \\ \nu = \dfrac{4\pi T r_o}{M v_o} . \end{cases}$$

The initial conditions (I-3) and (I-11) become

$$(4) \qquad r(0) = 1, \quad z(0) = 0, \quad \dot{r}(0) = \sin\alpha, \quad \dot{z}(0) = \cos\alpha.$$

If gravity effects are neglected, the problem can be solved in closed form. If $\Delta p \neq 0$, the solution involves elliptic functions and its discussion is too involved to be considered here. We shall therefore consider only the case of a vented jet ($\Delta p = 0$, i.e. $\eta = 0$), but will keep the gravity terms.

Multiply equations (2) scalarly with $\dot{\vec{x}}$. The result is

$$(5) \qquad \ddot{\vec{x}} \cdot \dot{\vec{x}} = \frac{1}{2}(|\dot{x}|^2)^{\bullet} = \mu \dot{z} ,$$

and on integrating and using (4),

(6) $$|\dot{x}|^2 = \dot{r}^2 + \dot{z}^2 = 1 + 2\mu z.$$

From this we get the inequality $|\dot{z}| \leq \sqrt{1 + 2\mu z}$, and as a consequence, the inequality

(7) $$|\dot{z}| \leq \sqrt{1 + 2\mu z} \leq 1 + \mu t.$$

This shows that for finite t, r, $\dot{r}$, z, and $\dot{z}$ remain bounded. Now, observing that

$$[(\dot{r}\ddot{z} - \dot{z}\ddot{r})|\dot{x}|^{-3}r + \dot{z}|\dot{x}|^{-1}]\dot{r} = (\dot{z}|\dot{x}|^{-1}r)^{\bullet} = \left(\frac{r\dot{z}}{\sqrt{1 + 2\mu z}}\right)^{\bullet}$$

the $\vec{b}$ component of equation (2) can be integrated yielding

(8) $$\left(1 - \frac{\nu r}{\sqrt{1 + 2\mu z}}\right)\dot{z} = \mu t + (1 - \nu) \cos \alpha.$$

The remainder of our considerations depend on whether $\nu < 1$ or $\nu > 1$, and we consider the case $\nu < 1$ first. In this case the right hand side of (8) remains positive, and since $\dot{z}$ is bounded, neither of the factors on the left hand side can vanish. Consequently, they preserve their signs at $t = 0$, and therefore

(9) $$\dot{z} > 0, \quad 1 - \frac{\nu r}{\sqrt{1 + 2\mu z}} > 0.$$

Incidentally, using (8) and inequality (7) we now get

$$\mu t + (1 - \nu)\cos \alpha = \left(1 - \frac{\nu r}{\sqrt{1 + 2\mu z}}\right)z \leq \sqrt{1 + 2\mu z} - \nu r$$

$$\leq 1 + \mu t - \nu r,$$

hence

(10) $$r \leq \frac{1}{\nu}\,[1 - (1 - \nu) \cos \alpha].$$

If $\nu = 1$, inequalities (9) still hold for $t > 0$, hence (10) holds also. We now have $r \leq 1$, and since this contradicts

the initial conditions $r(0) = 1$, $\dot{r}(0) = \sin \alpha > 0$, the problem has no solution for $\nu = 1$.

Since $\dot{z} > 0$, z can be chosen as the independent variable, and we consider r as a function of z. The equation that r satisfies can be obtained as follows. We compute $\ddot{z}$ in terms of $\dot{r}$, $\dot{z}$ and $\ddot{r}$ from equation (5) and substitute in the $\vec{a}$ component of (2) (which does not involve t explicitly). On solving for $\ddot{r}$, we thus get $\ddot{r}$ and therefore $\ddot{z}$ in terms of $r, z, \dot{r}, \dot{z}$. Now differentiating the relation $\dot{r} = r'\dot{z}$ (primes denoting differentiation with respect to z) we obtain r'' in terms of $\dot{z}, \ddot{r}, \ddot{z}$, hence in terms of $r, z, \dot{r}, \dot{z}$. Finally, $\dot{r}, \dot{z}$ are eliminated by using $\dot{z} = \sqrt{\frac{1 + 2\mu z}{1 + r'^2}}$ obtained from (6), and $\dot{r} = r'\dot{z}$. The result of these manipulations is

$$(11) \qquad r'' = - \frac{\mu r' + \nu\sqrt{1 + 2\mu z}}{1 - \frac{\nu r}{\sqrt{1 + 2\mu z}}} \cdot \frac{1 + r'^2}{1 + 2\mu z}$$

with the initial conditions

$$(12) \qquad r(0) = 1, \quad r'(0) = \tan \alpha.$$

Equation (11) can be integrated explicitly in the case when $\mu = 0$. We get then

$$\frac{r'r''}{1 + r'^2} = (\log \sqrt{1 + r'^2})' = \frac{-\nu r'}{1 - \nu r}$$

and on integrating,

$$(13) \qquad \sqrt{1 + r'^2} \cos \alpha = \left| \frac{1 - \nu r}{1 - \nu} \right| .$$

Equation (13) determines $|r'|$ and not r' itself. However, in view of (9), $r'' < 0$ for all z (if $\mu = 0$) and therefore r' decreases for all z. With this information, the integration can now be completed, and one obtains

(14) $$r = \frac{1}{\nu} - c \cosh \left(k - \frac{z}{c}\right)$$

where

(15) $$c = \frac{(1-\nu) \cos \alpha}{\nu}, \quad k = \cosh^{-1} \sec \alpha.$$

Equation (14) shows that r increases to a maximum $\bar{r}$

(16) $$\bar{r} = \frac{1}{\nu} - c = \frac{1}{\nu} [1 - (1-\nu) \cos \alpha]$$

attained at

(17) $$z = \bar{z} = ck$$

and then decreases to zero attained at

(18) $$z = \hat{z} = c(k + k')$$

where

(19) $$k' = \cosh^{-1} \sec \beta, \quad \cos \beta = (1-\nu) \cos \alpha.$$

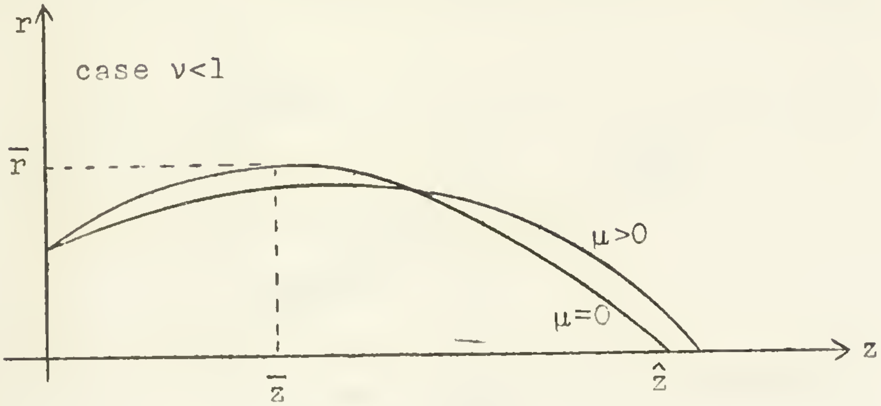

Figure 2

The behavior of $r(z)$ is not materially changed when $\mu > 0$. Indeed, in view of (9) r'' remains negative as long as r' is positive. Hence r' decreases to zero and then becomes negative. Since at $z = 0$, r'' has now a lower value than in the case $\mu = 0$, the curve initially runs below the one obtained before; see fig. 2. As r' decreases

further, r'' will eventually get closer to zero (or even change its sign) resulting in a flattening of the curve. However, r'' need not change sign before r becomes zero (that is, before the jet crosses the axis).

In case $\nu > 1$, the right hand term of equation (8) is initially negative but changes sign at the critical time $t_o = \frac{(\nu-1)\cos\alpha}{\mu}$ and remains positive for $t > t_o$. Until this time the two factors on the left hand side of (8) preserve their signs and we now have

$$(20) \qquad \dot{z} > 0, \quad \frac{\nu r}{\sqrt{1 + 2\mu z}} - 1 > 0 \quad \text{for} \quad t < t_o.$$

Hence, for $t < t_o$, z can be chosen again as independent variable, and in the same manner one arrives at equation (11) for $r(z)$, which shows that r'' remains positive and r' increases. If $\mu = 0$, one obtains thus the same result for $r(z)$ as equation (14) except that in the present case $c < 0$ and r increases exponentially. See fig. 3.

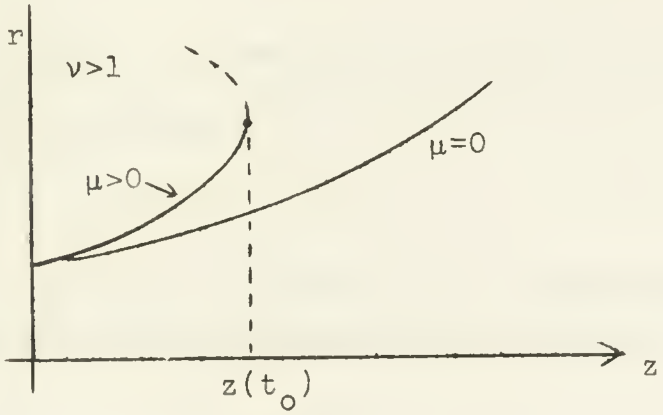

Figure 3

If the gravity term is included, the behaviour of $r(z)$ changes appreciably. To see this, we observe that at least one of the factors on the left hand side of equation (8) must vanish at $t = t_o$. Consider one of these, namely, $\nu r - \sqrt{1 + 2\mu z} = f(z)$. For $t = 0$, (i.e. $z = 0$), $f(0) = \nu - 1 > 0$, $f'(0) = \nu \tan\alpha - \mu$, which we assume positive,

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

whereas $f''(z) = \nu r'' + \mu^2(1 + 2\mu z)^{-3/2} > 0$ for $0 \leq t < t_o$ since $r'' > 0$. Hence $f'(z) > 0$ for $0 \leq t \leq t_o$ so that $f(z)$ increases, hence cannot vanish. We conclude then that $\dot{z}(t_o) = 0$, and since then $\dot{r}^2 = 1 + 2\mu z > 0$, r' is infinite and so is r''. The solution cannot be continued beyond t_o, for then z would have to decrease.

Corrections due to gravity terms can be estimated in the case $\nu < 1$. Denote by $r(z, \mu)$ the solution of (11) and assume that

$$(21) \qquad r(z, \mu) \sim r(z) + \mu R(z)$$

where $r(z)$ is the zero order solution given by (14). Substitute in (11) and expand all terms in power series in μ, neglecting terms of order two or higher. One thus obtains an equation for $R(z)$,

$$(22) \qquad c^2 R'' + 2c \tanh\left(k - \frac{z}{c}\right) R' + R = \frac{1}{\nu}\left[z - c \sinh\left(k - \frac{z}{c}\right) \cosh\left(k - \frac{z}{c}\right)\right]$$

with the initial conditions

$$R(0) = R'(0) = 0.$$

Since $\sinh(k - \frac{z}{c})$ is a solution of the associated homogeneous equation, (22) can be solved by quadratures and the result is

$$(23)\ R(z) = \frac{1}{\nu}\left[z - c \sinh\left(k - \frac{z}{c}\right)\cosh\left(k - \frac{z}{c}\right) + 2c \sec \alpha \sinh\left(k - \frac{z}{c}\right) - \tan \alpha\left(z \sinh\left(k - \frac{z}{c}\right) + c \cosh\left(k - \frac{z}{c}\right)\right)\right].$$

The maximum of $r(z, \mu)$ can be estimated by evaluating it at $z = \bar{z}$. We get

$$(24) \qquad \bar{r}(\mu) \sim r(\bar{z}) + \mu R(\bar{z}) = \frac{1}{\nu} - c - \frac{\mu c}{\nu}(\tan \alpha - k).$$

The point where $r(z, \mu)$ vanishes can be estimated by solving $r(z, \mu) = 0$ by Newton's method. We thus get

$$\hat{z}(\mu) = \hat{z} - \mu \frac{R(\hat{z})}{r'(\hat{z}) + \mu R'(\hat{z})} ,$$

or explicitly

$$\hat{z}(\mu) = c \left\{ k + k' + \frac{(k+k')\cos(\beta-\alpha)+\sec\beta \sin(\beta-\alpha)-2\sin\beta}{\frac{\nu}{\mu}\sin\beta\cos\alpha - (k+k')\sin\alpha - \frac{2\nu}{1-\nu}} \right\} \tag{25}$$

In a typical case, $\alpha = 30^\circ$, $\nu = .06$ and $\mu = .004$. We get

$$\bar{r} = 3.100, \quad \mu R(\bar{z}) = -.025$$

$$\hat{z} = 16.12, \quad -\mu \frac{R(\hat{z})}{r'(\hat{z}) + \mu R'(\hat{z})} = .275.$$

Corrections due to gravity become important when r is small. In the case considered, $r(2\bar{z}) = 1$, whereas $\mu R(2\bar{z}) = +.121$, or 12°/o.

III. Comparison with experimental results.

Experiments on jets produced by conical nozzles were performed recently by J. S. Derr, Jr. at Camp Detrick, Frederick, Maryland, who supplied the data used below. The nozzle used had the following characteristics:

Radius at orifice, $r_o = .508$
Nozzle area perpendicular to exit section, $A = .0631$
Nozzle angle, $\alpha = 30^\circ$
Nozzle thickness, $h_o = .0203$.

The surface tension of the liquid was assumed to be $T = 71$, density, $\rho = 1$. Three flows were observed with mass flux $M = 21.5$, 18.4, 14.2 respectively. All of the above data are expressed in CGS units.

Since $v_o = \frac{M}{\rho A}$, one gets for ν

(1) $$\nu = \frac{4\pi T_o r_o A}{M^2} = 28.6\ M^{-2} ,$$

using the above data. Hence all the three flows considered fall in the case $\nu < 1$. If one assumes that the initial jet angle is equal to the nozzle angle, the sole knowledge of mass flux and nozzle characteristics are sufficient to determine the theoretical curves.

Using the uncorrected formulae (III-14-19) we now get the following comparison table. (Lengths are expressed in units of r_o.)

	M	21.5	18.4	14.2
	ν	.0618	.0854	.1418
$\bar{r}$	Computed	3.05	2.44	1.81
	Observed	3.94	3.50	2.44
	Error	-25%	-31%	-26%
$\bar{z}$	Computed	7.23	5.10	2.88
	Observed	8.00	7.39	3.78
	Error	-10%	-32%	-24%
$\hat{z}$	Computed	16.5	11.70	7.12
	Observed	19.6	15.87	8.63
	Error	-16%	-26%	-18%

The discrepancy between the data thus computed and the observed data is too big to be accounted for by corrections due to the gravity terms. The cause of this discrepancy lies in that there must be some sticking of the liquid to

the nozzle causing the initial jet angle to be different from that of the nozzle. Indeed, a further scrutiny of the experimental data shows that this angle is larger than the nozzle angle and increases when the mass flux decreases, and, as formulae (III - 14-19) show, $\bar{r}$, $\bar{z}$, $\hat{z}$ are sensitive to the variation of this angle.

To determine a theoretical curve, α, ν, and μ must be known. These are related by the condition

$$(2) \qquad \frac{\nu}{\mu} = \frac{2T}{\rho g r_o h_o \cos \bar{\alpha}} \cos \alpha$$

obtained from (I-1) and (II-3), involving only static quantities. Here we denoted by $\bar{\alpha}$ the nozzle angle (equal to 30° in the present case) which now is assumed different from the initial jet angle α. Since α cannot be measured accurately, it is preferred to determine ν, μ, and α by requiring that predicted values of $\bar{r}$ and $\hat{z}$ agree with the experimental values. Using the equations (II-24) and (II-25) for $\bar{r}$ and $\hat{z}$, we thus obtain together with (2) three equations for ν, μ, and α. For the three flows considered these yield

Case (M)	α°	ν	μ
1 (21.5)	34.3	.0559	.00418
2 (18.4)	36.3	.0720	.00553
3 (14.2)	39.8	.1386	.01117

$r(z)$ computed with these data for α, ν, μ are graphed on fig. 4, showing an excellent agreement with the experimental data. In case 2 and 3 the discrepancy never exceeds $2^\circ/o$ of measured data. In case 1 there is, at some

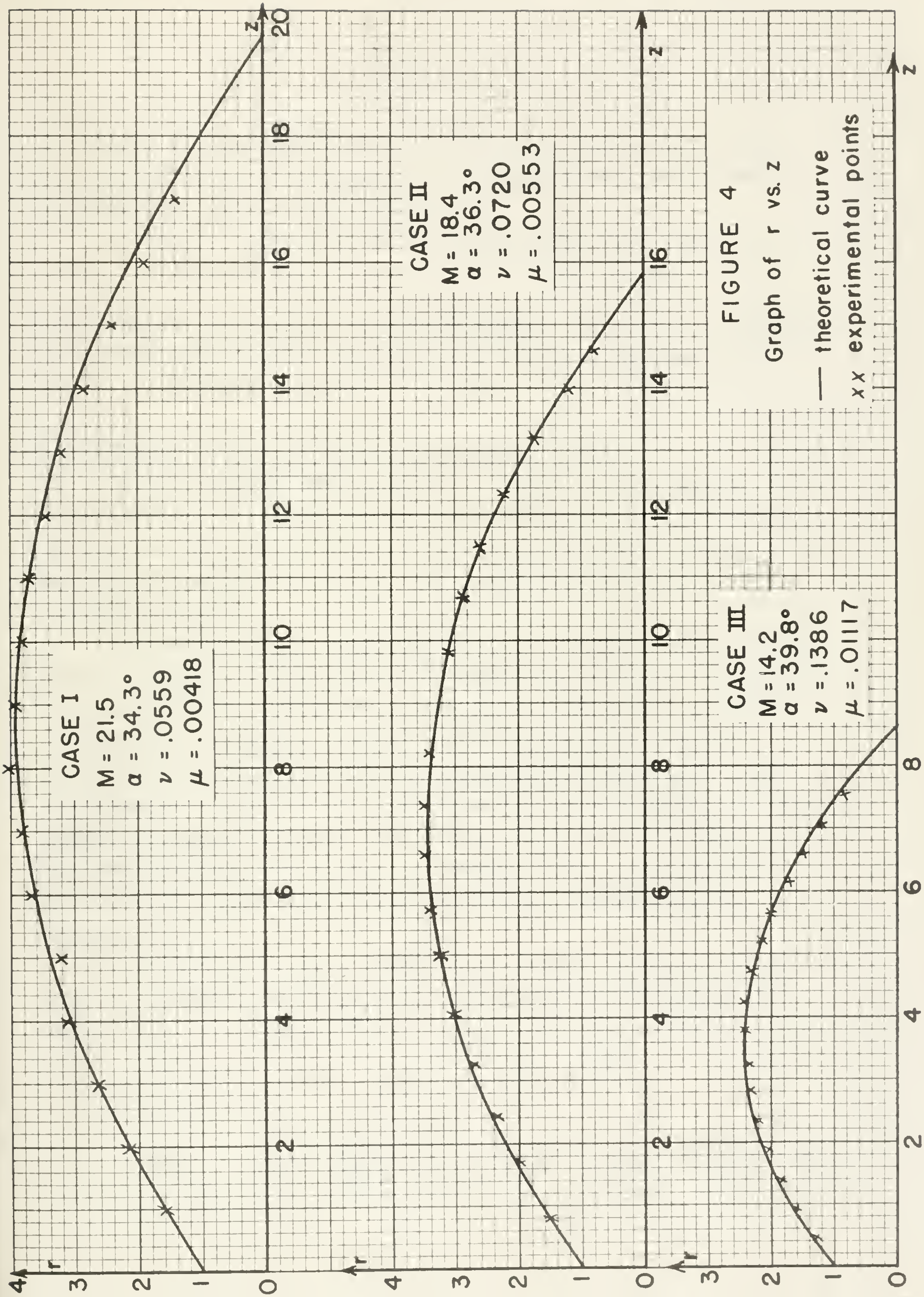

FIGURE 4

Graph of r vs. z

points, a divergence of about $10^{\circ}/o$, but this is undoubtedly due to a poor extrapolation of measured data which was necessary to obtain the value of $\hat{z}$.

As to the case $\nu > 1$, no experimental data are available. As formula (II-3) shows, $\nu > 1$ when $\frac{1}{2}\rho v_o^2 < \frac{T}{h_o}$, which would occur when $v_o < 83.6$ cm/sec or $M < 5.28$ g/sec. It is doubtful whether in this case the thin jet assumption leads to coherent results.

IV. Theory of oscillations of unsteady jets.

If the initial jet conditions, $\vec{x}(\tau,0)$ and $\vec{x}_t(\tau,0)$, (see section I), deviate from the equilibrium jet discussed in section II, an oscillatory motion is expected. The mechanism leading to such a motion was already discussed in the introduction.

In general it is very difficult to solve the initial value problem for the system of equations (I-6). Fortunately such a solution is not even desired, for it is not natural to prescribe specific initial conditions beyond the fact that they deviate from the equilibrium configuration. An approximate theory is obtained from the consideration of the *energy equation* obtained by dotting equation (I-6) with $\vec{v}$ and integrating over the jet envelope (i.e., with respect to τ) at a fixed time. The interval of integration is $(\tau_o(t), t)$.

We consider various terms separately. Denoting by v_n and v_s normal and tangential components of velocity, we have

$$(1)\quad M\int_{\tau_o(t)}^{t} \vec{x}_{tt}\cdot\vec{v}\,d\tau = \frac{1}{2}M\int_{\tau_o}^{t}(v_n{}^2 + v_s{}^2)d\tau$$

$$= \frac{1}{2}M\int_{\tau_o}^{t} v_s{}^2 d\tau + \frac{d}{dt}\left(\frac{1}{2}M\int_{\tau_o}^{t} v_n{}^2 d\tau\right) + \frac{1}{2}M\left(v_{nf}^2\,\frac{d\tau_o(t)}{dt} - v_{no}^2\right).$$

Here, v_{no} is the normal component of the initial jet velocity and v_{nf} is the normal component of the velocity at the tip of the jet. Next, it is easily verified that

$$(2) \qquad 2\pi \int_{\tau_o(t)}^{t} r|x_\tau| H\vec{q} \cdot \vec{v} \, d\tau = - \frac{dS(t)}{dt}$$

where S is the area of the streamsurface,

$$S = 2\pi \int_{\tau_o(t)}^{t} r|x_\tau| \, d\tau,$$

and that

$$(3) \qquad -2\pi \int_{\tau_o(t)}^{t} r|x_\tau| \vec{q} \cdot \vec{v} \, d\tau = \frac{dV(t)}{dt} ,$$

where V(t) is the volume enclosed by the jet, see equation (I-10). If we denote by G(v) the internal energy of the air enclosed by the jet,

$$(4) \qquad G(V) = \int_{V}^{\infty} P(\sigma) d\sigma \quad ,$$

the two last terms in (I-6) contribute on integration

$$- \frac{d}{dt} (+ G + pV + 2TS).$$

Combining the above we thus get on integrating (I-6),

$$(5) \qquad \frac{d}{dt} \left[\frac{1}{2} M \int_{\tau_o(t)}^{t} v_n{}^2 d\tau + G(V) + pV + 2TS\right]$$
$$+ \frac{1}{2} M\left(v_{nf}^2 \frac{d\tau_o(t)}{dt} - v_{no}^2\right) + M \int_{\tau_o(t)}^{t} \left(\frac{1}{2} v_s^2 - gz\right)_t d\tau$$
$$= 0.$$

The first term in the bracket can be viewed as the kinetic energy of the normal motion, whereas the others - as the potential energy.

If the jet were in equilibrium, the normal component of the fluid velocity would vanish and the tangential component would satisfy

$$\frac{1}{2} v_s^2 - gz = 0,$$

according to equation (II-6) (written in original units). Furthermore, since v_s would be practically constant and equal to v_o, the equation of the jet could be written in the form

$$(6) \qquad r = f(z), \quad f(0) = r_o,$$

with the parametric representation

$$(7) \qquad z = z(\tau), \quad r = f(z(\tau)), \quad r_\tau^2 + z_\tau^2 = v_o^2 .$$

Since our object is primarily the study of deviations from the equilibrium jet, it is plausible to assume that at each time the jet is approximately in equilibrium, so that it has a representation of the form (6) and (7), now depending on time through some parameter $R(t)$. That is

$$(6') \qquad r = f(z,R), \quad f(0,R) = r_o = \text{a constant in time.}$$

In this approximation the last term in (5) vanishes. The normal velocity component is

$$(8) \qquad v_n = \frac{f_R \dot{R}}{\sqrt{1 + f'^2}}$$

and from (6') one gets $v_{no} = 0$. Furthermore, if the parametric representation (7) is so chosen that $z(0) = 0$, the time that a particle takes to flow through the jet, i.e., $t - \tau_o(t)$, is equal to the value of τ for which $f(z,R)$ vanishes. This value depends on R and will be denoted by $\tau(R)$. Hence,

$$(9) \qquad \frac{d\tau_o(t)}{dt} = \frac{d}{dt}(t - \tau(R)) = 1 - \tau'(R)\dot{R}.$$

Likewise, the final value of z, i.e. $\tilde{z}(\tau(R))$ will depend on R. Using the notation

$$(10) \qquad M \int_0^{\tau(R)} \frac{f_R^{\,2}}{1 + f'^2} \, d\tau = m(R),$$

$$(11) \qquad G(v) + pV + 2TS = q(R),$$

$$(12) \qquad \frac{1}{2} M \frac{f_R^2(z(\tau(R)), R)}{1 + f'^2(z(\tau(R)),R)} = \frac{1}{2} M\left(\frac{v_{nf}}{\dot{R}}\right)^2 = s(R),$$

equation (5) now becomes an equation for $R(t)$, namely

$$(13) \qquad \frac{d}{dt} \left(\frac{1}{2} m\dot{R}^2 + q\right) + s(1 - \tau'\dot{R})\dot{R}^2 = 0.$$

Equation (13) for $R(t)$ actually describes the response of a thin jet which at each time is constrained to take the form prescribed by (6'), and which is traversed by an incompressible fluid with velocity v_o. The description will be satisfactory if equation (13) has an equilibrium solution $R = R_o$ = constant, and for R_o, $f(z,R_o)$ is the equilibrium jet found in section II, whereas for R differing slightly from R_o, $f(z,R)$ differs little from the equilibrium shape.

The manifold of solutions of (13) can be discussed without reference to the specific choice of $f(z,R)$, except for some mild assumptions on continuity and differentiability of functions appearing in (13). We need not be concerned with these here, but shall assume, in accordance with further applications that there exists a unique value R_o for which

$$(14) \qquad \begin{cases} q'(R_o) = 0, \\ 4mq'' - s^2 > 0. \end{cases}$$

Carrying out the differentiation in (13) we get, on dividing by $\dot{R}$,

$$(15) \qquad m\ddot{R} + s\dot{R} + \left(\frac{m'}{2} - s\tau'\right)\dot{R}^2 + q' = 0.$$

It is immediately verified that $R = R_o$ is a solution of (15) and we shall now show that any solution of (15) which gets close to R_o performs damped oscillations about R_o.

In a t interval where $R(t)$ is monotonic, a unique inverse $t(R)$ is defined. Introducing $u(R)$ defined by

$$(16) \qquad u(R) = \dot{R}(t(R))$$

one gets from (15) a first order equation for u,

$$(17) \qquad u' = -\frac{q' + su + \left(\frac{m'}{2} - s\tau'\right)u^2}{mu} \quad .$$

The point $R = R_o$, $u = 0$ is a singular point of this equation, and an easy verification shows that it is, provided that assumption (14) holds, a vortex point. Consequently, the solutions of (17) near $R = R_o$, $u = 0$ spiral about this point, see fig. 5. The reinterpretation of this "phase space" diagram in the "physical space" leads to damped oscillatory motion.

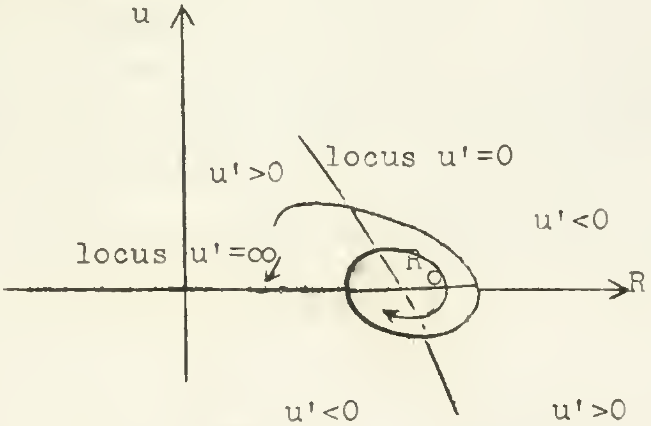

Figure 5

Since R satisfies a non-linear equation, its "period" (time elapsed between two successive maxima) and its "half time" depend on initial data. One obtains an estimate of

the period and the half time by assuming a motion of small amplitude and linearizing the differential equation. Setting $R = R_o + \varepsilon\bar{R}$, and linearizing (15) one now gets

$$(18) \qquad m(R_o)\ddot{\bar{R}} + s(R_o)\dot{\bar{R}} + q''(R_o)\bar{R} = 0.$$

Equation (18) has solutions of the form $\exp(-\kappa + \Delta i)t$, with period

$$(19) \qquad T = \frac{2\pi}{\Delta} = \frac{2\pi}{\sqrt{\frac{q''}{m} - \left(\frac{s}{2m}\right)^2}} ,$$

and half-time

$$(20) \qquad \Theta = \frac{1}{\kappa} = \frac{2m}{s}$$

(in the above, m, s, and q'' are evaluated at $R = R_o$).

The above considerations indicate that the jet will perform a damped oscillatory motion, provided that the important assumption (14) is satisfied. Actual estimates of T and Θ depend on the choice of $f(z,R)$. Since our theory is only approximate there is no criterion for the best choice of f. Undoubtedly a satisfactory result would be obtained by choosing for f the function

$$(21) \qquad r = f(z,R) \equiv \frac{r_o}{R} - \frac{(1 - R)r_o \cos\alpha}{R}\cosh\left[\cosh^{-1}\sec\alpha - \frac{Rz}{(1 - R)r_o\cos\alpha}\right]$$

which for $R = \nu$ reduces to the equilibrium solution for the jet envelope derived in section II (eq. II-14). This function, which is obtained from (II-14) by replacing ν by R has also the advantage that its slope at $z = 0$ is $\tan\alpha$, (no matter what R is), so that it represents at each instant an equilibrium jet corresponding to some mass

flow, emerging with the same angle.

To use (21) for $f(z,R)$ would lead, however, to complicated calculations. It is felt that not much is lost by using a "jet" shape which is conveniently simple, but either has an equilibrium area equal to that of a jet actually considered, or encloses the same volume in equilibrium. For the sake of simplicity we therefore assume that the jet is a sphere of radius $R(t)$ attached to the nozzle. See Fig. 6.

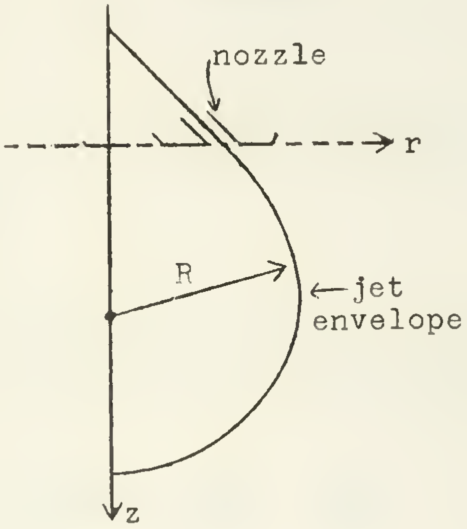

Figure 6

Its equation is

$$(22) \qquad r = f(z,R) = \sqrt{R^2 - (z - C)^2}\ , \text{ where } C = \sqrt{R^2 - r_o^2}.$$

The desired parametric representation is

$$\begin{cases} z = C - R \sin[\sin^{-1} \frac{C}{R} - \frac{v_o}{R} \tau] \\ r = R \cos[\sin^{-1} \frac{C}{R} - \frac{v_o}{R} \tau], \end{cases}$$

and

$$\begin{cases} \tau(R) = \frac{R}{v_o} (\frac{\pi}{2} + \sin^{-1} \frac{C}{R}), \\ z(\tau(R)) = R + C. \end{cases}$$

[illegible]

[illegible]

Since

$$V' = \frac{d}{dR}\left(\pi \int_0^{R+C} f^2 dz\right) = \frac{\pi R(R+C)^2}{C} ,$$

$$S' = \frac{d}{dR}\left(2\pi \int_0^{R+C} f \sqrt{1+f'^2}\, dz\right) = \frac{2\pi(R+C)^2}{C} ,$$

one obtains

$$q'(R) = -PV' + pV' + 2TS' = \frac{\pi R(R+C)^2}{C}\left[-P(V) + p + \frac{4T}{R}\right]$$

using the definition (11). The equilibrium radius R_o is therefore determined as the solution of equation

$$P(V) = p + \frac{4T}{R} .$$

Assuming the adiabatic relation for the air enclosed by the jet, $P = kV^{-\gamma}$, we now get for the equilibrium value of q'',

$$(23) \qquad q''(R_o) = \frac{\pi R(R+C)^2}{C}\left[\frac{\gamma V'}{V}\left(p + \frac{4T}{R}\right) - \frac{4T}{R^2}\right]$$

$$= \frac{\pi R(R+C)^2}{C}\left[\frac{3\gamma R(R+C)}{C[R(R+C)+r_o^2]}\left(p + \frac{4T}{R}\right) - \frac{4T}{R^2}\right],$$

evaluated for $R = R_o$. Using the definitions (10) and (12) for $m(R_o)$ and $s(R_o)$ we find

$$(24) \qquad m = \frac{MR}{v_o}\left[\frac{3r_o}{2C} + \left(1 + \frac{R^2}{2C^2}\right)\left(\frac{\pi}{2} + \sin^{-1}\frac{C}{R}\right)\right],$$

$$(25) \qquad s = \frac{M}{2}\left(\frac{R+C}{C}\right)^2.$$

Even the simple model (22) leads to rather cumbersome formulae for the period and the half-time. A further simplification is obtained on expanding all formulae in powers of r_o/R. We get

$$(23') \qquad q''(R_o) = 12\pi\gamma pR\left(1 + \frac{5}{16}\left(\frac{r_o}{R}\right)^4 + \dots\right) + 16\pi(3\gamma-1)T\left(1 + \frac{15\gamma-1}{16(3\gamma-1)}\left(\frac{r_o}{R}\right)^4 + \dots\right),$$

$$(24') \qquad m = \frac{3\pi MR}{2v_o}\left(1 + \frac{1}{3}\left(\frac{r_o}{R}\right)^2 + \dots\right),$$

$$(25') \qquad s = 2M\left(1 + \frac{1}{2}\left(\frac{r_o}{R}\right)^2 + \dots\right).$$

In the application to be considered, $\frac{r_o}{R} < \frac{1}{3}$. Hence, omitting the terms involving r_o will result in an error of less than 0.5°/o in case of q'', and of about 5°/o in the evaluation of m and s. Incidentally, setting $r_o = 0$ amounts to taking in place of the "jet" described by (22), a full sphere touching the nozzle at one point. Setting $r_o = 0$ and substituting (23')-(25') in (19) and (20), we get the approximate formulae

$$(26) \qquad T = 2\pi\left[\frac{2v_o}{3\pi R}\left(12\gamma\pi\frac{pR}{M} + 16\pi(3\gamma-1)\frac{T}{M} - \frac{2v_o}{3\pi R}\right)\right]^{-1/2},$$

$$(27) \qquad \theta = \frac{3\pi R}{2v_o}.$$

As an application we consider the three cases discussed in section III. As was already indicated, we could require that either the area of the sphere is equal to the area of the actual jet, or the volume of the sphere is equal to the volume enclosed by the actual jet. Each of these criteria will lead to a slightly different value for the equilibrium radius. We use R_o for the radius determined by the first of these and $\bar{R}_o$ for the radius determined by the second. One gets, in CGS units,

Case	M	v_o	V	S	R_o	$\bar{R}_o$
1	21.5	341	68.5	92.4	2.73	2.54
2	18.4	292	41.0	64.6	2.29	2.14
3	14.2	225	12.4	26.7	1.50	1.43

Using $\bar{R}_o$, in (26) and (27), we finally get

Case	θ	T	
		$p=1$ atmos.	$p=0$
1	.035	$.45 \times 10^{-3}$	.053
2	.034	$.45 \times 10^{-3}$	.048
3	.030	$.45 \times 10^{-3}$	.039

When $p > .001$ atmos., the last two terms in the bracket of (26) are unimportant, and the formula for the period reduces to

$$T \sim 1.9 \sqrt{\frac{A\rho}{p}} \tag{28}$$

where A is the area of the exit section of the nozzle. When the pressure is below this value, formula (28) becomes inapplicable, but as the table shows, we get about 20 to 25 cycles per second. The half-time, on the other hand, is independent of pressure, and is of the order of .03 sec.

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